

Some Results on Bent-Negabent Boolean Functions over Finite Fields*

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Abstract

We consider negabent Boolean functions that have Trace representation. We completely characterize quadratic negabent monomial functions. We show the relation between negabent functions and bent functions via a quadratic function. Using this characterization, we give infinite classes of bent-negabent Boolean functions over the finite field \mathbb{F}_{2^n} , with the maximum possible degree, $\frac{n}{2}$. These are the first ever constructions of negabent functions with trace representation that have optimal degree.

Keywords: Negabent function, bent function, quadratic Boolean function, Maiorana-McFarland function, permutation, complete mapping polynomial.

1 Introduction

Hadamard-Walsh transform is an important tool in characterizing Boolean functions. For example, many cryptographic properties can be analyzed by the Hadamard-Walsh transform. A function on even number of variables that has the maximum possible distance from the affine functions is called a bent function. These functions have equal absolute spectral values under the Hadamard-Walsh transform and was first introduced by Rothaus [Rot76]. It is natural to investigate the spectral values of Boolean functions under some other Fourier transform. In 2007, Parker and Pott [PP07],

*This is a sufficiently revised and extended version of the paper [Sar12]. Section 5 is a completely new contribution.

considered the nega-Hadamard transform. and introduced negabent functions. These functions have equal absolute spectral values under the nega-Hadamard transform. The periodic autocorrelation values of a bent function are all zero. The negaperiodic autocorrelation value of a Boolean function under the nega-Hadamard transform is the analogue of the periodic autocorrelation value. The negaperiodic autocorrelation values are all zero for a negabent function. These properties of a negabent function motivate us to study it further. Negabent functions which are also bent are interesting as they have extreme properties in terms of two different Fourier transforms.

Results on negabent functions can be found in [PP07, SPP08, Par00, RP05, SGC⁺12, Sar09]. As an example, the 6-variable function $x_4(x_1x_2 \oplus x_2x_3 \oplus x_1 \oplus x_2) \oplus x_5(x_1x_2 \oplus x_2x_3 + x_3) + x_6(x_1 \oplus x_3)$ is a cubic negabent function.

In [PP07, SPP08] some classes of Boolean functions which are both bent and negabent (bent-negabent) have been identified. In [SPP08], construction of negabent functions has been shown in the class of Maiorana-McFarland bent functions. It is interesting to note that all the affine functions (both odd and even variables) are negabent [PP07, Proposition 1]. In [Sar09], symmetric negabent functions have been characterized and shown to be all affine for both odd and even number of variables. The maximum degree of an n -variable bent-negabent function is $\frac{n}{2}$. Very recently construction of bent-negabent functions have been given in [SPT13] with the optimal degree.

In this paper, we characterize the negabent functions which are defined over finite fields, *i.e.*, functions with Trace representation.

Let \mathbb{F}_2^n be the vector space formed by the binary n -tuples and \mathbb{F}_{2^n} be the finite field with 2^n elements. For a set E , the set of non zero elements of E is denoted by E^* .

In [PP07], quadratic negabent Boolean functions defined over the vector space \mathbb{F}_2^n were characterized. Any quadratic Boolean function can be written as

$$\begin{aligned} g(x_1, \dots, x_n) &= \sum_{1 \leq i < j \leq n} q_{i,j} x_i x_j + \sum_{1 \leq i \leq n} l_i x_i + c \\ &= xQx^T + Lx^T + c, \end{aligned}$$

where $Q = (q_{i,j})$ is an upper triangular binary matrix, $L = (l_1, \dots, l_n)$ is a binary vector and $c \in \{0, 1\}$. Consider the binary symmetric matrix $B = Q + Q^T$ with the zero diagonal, which is the symplectic matrix corresponding to the quadratic function g . It is well known that a quadratic function g is

bent if and only if the corresponding matrix B has full rank. In [PP07], it was proved that a quadratic function g is negabent if and only if the matrix $B + I$ has full rank, where B is the corresponding symplectic matrix and I is the identity matrix.

In this paper, first we consider quadratic monomials defined over the field \mathbb{F}_{2^n} and is of the form

$$f : x \mapsto \text{Tr}_1^n(\lambda x^{2^k+1}), \quad \text{where } \lambda \in \mathbb{F}_{2^n}^*. \quad (1)$$

The λ 's for which f is bent is well known. We characterize those λ 's for which f is negabent. We also give characterization of λ for even n such that f is bent-negabent. The existence of quadratic bent-negabent functions is known [PP07]. We reprove the existence by simple counting argument and using the characterization of quadratic bent-negabent monomials.

We also study the negabent property of Maiorana-McFarland bent functions $f : \mathbb{F}_{2^t} \times \mathbb{F}_{2^t} \mapsto \mathbb{F}_2$ defined by $f(x, y) = \text{Tr}_1^t(x\pi(y) + h(y))$, where $\pi(y)$ is a permutation polynomial over \mathbb{F}_{2^t} and $h(y)$ is any polynomial over \mathbb{F}_{2^t} . We present a necessary and sufficient condition such that these functions are negabent. As a consequence, we show that when the permutation π is $x \mapsto x^{2^i}$, the bent function f is negabent if and only if $h(y)$ is bent. From this we show how a Maiorana-McFarland bent-negabent function of degree $\frac{n}{4}$ over \mathbb{F}_{2^n} can be obtained.

Then we show that given a bent function f over \mathbb{F}_{2^n} , it is possible to obtain a negabent function by adding a quadratic function, and vice versa. Using this result we are able to present infinite classes of bent-negabent functions having the optimal degree $\frac{n}{2}$.

2 Preliminary

An n -variable Boolean function is a mapping $f : \mathbb{F}_2^n \mapsto \mathbb{F}_2$. The Hamming weight of a binary string S is the number of 1's in S and it is denoted as $wt(S)$. An n -variable Boolean function f can be written as a function of x_1, \dots, x_n variables as follows,

$$f(x_1, x_2, \dots, x_n) = \bigoplus_{a=(a_1, \dots, a_n) \in \mathbb{F}_2^n} \mu_a \left(\prod_{i=1}^n x_i^{a_i} \right), \quad \text{where } \mu_a \in \mathbb{F}_2.$$

This is called the algebraic normal form (ANF) of f . The degree, $\deg(f)$, of f is defined as $\max_{a \in \mathbb{F}_2^n} \{wt(a) | \mu_a \neq 0\}$.

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ and $x = (x_1, \dots, x_n)$ be two vectors in \mathbb{F}_2^n and $\lambda \cdot x = \lambda_1 x_1 \oplus \dots \oplus \lambda_n x_n$. Then the Hadamard-Walsh transform value of f at λ is given by

$$\mathcal{H}_f(\lambda) = \frac{1}{2^{\frac{n}{2}}} \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) \oplus \lambda \cdot x}. \quad (2)$$

The function is called bent if $|\mathcal{H}_f(\lambda)| = 1$ for all $\lambda \in \mathbb{F}_2^n$. After the introduction of bent functions in [Rot76], there have been many constructions of bent functions, for instance, [Dil74], [Car93], [LHTK13], and references therein.

For $a \in \mathbb{F}_2^n$, the periodic autocorrelation value of f is computed as

$$\tau_a = \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) \oplus f(x \oplus a)}.$$

A Boolean function f is bent if and only if $\tau_a = 0$ for all $a \in \mathbb{F}_2^{n*}$.

The nega-Hadamard transform value of f at $\lambda \in \mathbb{F}_2^n$ is given by

$$\mathcal{N}_f(\lambda) = \frac{1}{2^{\frac{n}{2}}} \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) \oplus \lambda \cdot x} I^{wt(x)}, \quad (3)$$

where $I = \sqrt{-1}$, is the imaginary unit of the complex number. Note that $\mathcal{N}_f(\lambda)$ is complex valued.

The function $f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ is called *negabent* [PP07] if the magnitude of $\mathcal{N}_f(\lambda)$ is 1, i.e., $|\mathcal{N}_f(\lambda)| = 1$ for all $\lambda \in \mathbb{F}_2^n$. In the following theorem we state an alternate characterization of negabent functions in terms of their negaperiodic autocorrelation values which has been shown in [PP07, Theorem 2] and [SGC⁺12, Lemma 3].

Theorem 1. *A Boolean function f is negabent if and only if*

$$\sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) \oplus f(x \oplus y)} (-1)^{x \cdot y} = 0 \quad (4)$$

for all $y \in \mathbb{F}_2^{n*}$.

From this theorem, we see that the correlation values between the function $f(x)$ and $f(x \oplus y) \oplus y \cdot x$ are all zero for all $y \in \mathbb{F}_2^{n*}$. Moreover, for even number of variables, if a negabent function is also a bent function, then the correlation values of the function $f(x)$ and $f(x \oplus y)$ are also equal to zero for all $y \in \mathbb{F}_2^{n*}$. Therefore, the functions which are both bent and negabent are interesting to study. We call these functions *bent-negabent*. Bent functions can exist only on even number of variables and it has degree more than 1. However, all the affine functions are negabent [PP07] which tells that negabent functions exist for both even and odd number of variables.

3 Characterization of negabent functions over the finite field \mathbb{F}_{2^n}

Now we consider Boolean functions defined over the field \mathbb{F}_{2^n} and we characterize the negabent property of those functions. The vector space \mathbb{F}_2^n can be easily identified with the field \mathbb{F}_{2^n} by choosing a basis of \mathbb{F}_{2^n} over \mathbb{F}_2 . The function $Tr_1^n : \mathbb{F}_{2^n} \mapsto \mathbb{F}_2$ is defined as

$$Tr_1^n(x) = x + x^2 + \dots + x^{2^{n-1}}.$$

We denote Tr_1^n simply by Tr and “+” is the finite field addition. If we choose the basis $\{\alpha_1, \dots, \alpha_n\}$ to be self dual then it can be shown that $Tr(xy) = \sum_{i=1}^n x_i y_i$.

Henceforth, in this paper we choose the basis to be self dual. It is also notable that a linear function over \mathbb{F}_{2^n} is given by $\ell(x) = Tr(ax)$, $a \neq 0$.

Given a polynomial $F(x)$ over \mathbb{F}_{2^n} , we can get a Boolean function $f : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_2$ defined as $f(x) = Tr(F(x))$. The highest binary weight of the exponents of $F(x)$ is denoted as the algebraic degree of $F(x)$, then the degree of $f(x) = Tr(F(x))$ is equal to the algebraic degree of $F(x)$.

With the above discussions it is now clear that we can characterize negabent functions defined over the finite field as follows.

Proposition 1. *The function $f : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_2$ is negabent if and only if*

$$\sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x) + f(x+a) + Tr(ax)} = 0 \quad (5)$$

for all $a \in \mathbb{F}_{2^n}^*$.

It is known that a bent function $f : \mathbb{F}_{2^n} \mapsto \mathbb{F}_2$ is bent if and only if

$$\sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x) + f(x+a)} = 0 \quad (6)$$

for all $a \in \mathbb{F}_{2^n}^*$.

A function for which both (5) and (6) hold is a bent-negabent function.

An n -variable Boolean function ϕ is called balanced if its weight is 2^{n-1} . Note that $\sum_{x \in \mathbb{F}_{2^n}} (-1)^{\phi(x)} = 0$ if and only if ϕ is balanced. Therefore, if f is bent-negabent, then both $f(x) + f(x+a)$ and $f(x) + f(x+a) + Tr(ax)$ are balanced for all $a \in \mathbb{F}_{2^n}^*$.

3.1 Linear structure of negabent functions

As the sum (5) involves derivatives of f , *i.e.*, $f(x) + f(x+a)$, here we briefly discuss about the linear structures of negabent functions.

Definition 1. An $a \in \mathbb{F}_{2^n}^*$ is said to be a linear structure of a polynomial $F(x)$ over \mathbb{F}_{2^n} if the derivative $F(x) + F(x+a)$ is constant.

From (6), it is clear that a bent function can not have a linear structure. However, a negabent function can have a linear structure. In fact, if f is such that any $a \in \mathbb{F}_{2^n}^*$ is a linear structure, then f is negabent. As in that case, the term $f(x) + f(x+a) + ax$ is $c + ax$, for some constant c , which is affine, *i.e.*, balanced. This happens when $f(x)$ is an affine polynomial, which proves that if f is affine it is negabent.

A polynomial $F(x)$ over \mathbb{F}_{2^n} is called a complete mapping polynomial if both $F(x)$ and $F(x) + x$ are permutation polynomials. We use such permutation polynomials in our constructions.

3.2 Quadratic negabent monomial functions

We consider quadratic monomials and characterize when they are negabent.

Proposition 2. Let $f : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_2$ be a quadratic function of the form

$$f(x) = \text{Tr}(\lambda x^{2^k+1}).$$

Then f is negabent if and only if

$$\lambda^{2^{n-k}} a^{2^{n-k}} + \lambda a^{2^k} + a \neq 0 \quad (7)$$

for all $a \in \mathbb{F}_{2^n}^*$.

Proof. From (5) we know that the function f is negabent if and only if for all $a \in \mathbb{F}_{2^n}^*$,

$$\begin{aligned} \sum_{x \in \mathbb{F}_{2^n}} (-1)^{\text{Tr}(\lambda x^{2^k+1}) + \text{Tr}(\lambda(x+a)^{2^k+1}) + \text{Tr}(ax)} &= 0, \\ \text{i.e., } \sum_{x \in \mathbb{F}_{2^n}} (-1)^{\text{Tr}((\lambda^{2^{n-k}} a^{2^{n-k}} + \lambda a^{2^k} + a)x)} &= 0. \end{aligned} \quad (8)$$

Note that (8) is true if and only if

$$\lambda^{2^{n-k}} a^{2^{n-k}} + \lambda a^{2^k} + a \neq 0,$$

for all $a \in \mathbb{F}_{2^n}^*$. Hence the result. □

Proposition 3. *The quadratic function $f : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_2$ of the form $f(x) = \text{Tr}(\lambda x^{2^k+1})$ is negabent if and only if*

$$P(x) = \lambda^{2^{n-k}} x^{2^{n-k}} + \lambda x^{2^k} + x$$

is a permutation polynomial over \mathbb{F}_{2^n} .

Proof. From Proposition 2, we know that f is negabent if and only if

$$\lambda^{2^{n-k}} a^{2^{n-k}} + \lambda a^{2^k} + a \neq 0$$

for all $a \in \mathbb{F}_{2^n}^*$. That means for such $\lambda \in \mathbb{F}_{2^n}^*$, $P(x) = \lambda^{2^{n-k}} x^{2^{n-k}} + \lambda x^{2^k} + x$ has no non zero root in \mathbb{F}_{2^n} . Note that $P(x)$ is a linearized polynomial and linearized polynomial is permutation if and only if it has no non zero root. Hence the result. \square

The polynomial $P(x)$ is a permutation over \mathbb{F}_{2^n} if and only if the polynomial $P(x)^{2^k}$ is a permutation over \mathbb{F}_{2^n} . Therefore, $\text{Tr}(\lambda x^{2^k+1})$ is a negabent if and only if $\lambda x + \lambda^{2^k} x^{2^{2k}} + x^{2^k}$ is a permutation. There are some results on the number of solutions of the polynomial $\lambda^{2^k} x^{2^{2k}} + x^{2^k} + \lambda x$ in [HK10].

Let us point out some results related to the equation

$$\lambda^{2^k} x^{2^{2k}} + x^{2^k} + \lambda x = 0$$

which has been extensively studied in [HK10]. Let $\gcd(k, n) = d \geq 1$ and $n = td$ for $t > 1$. A particular sequence of polynomials over \mathbb{F}_{2^n} is introduced as follows.

$$\begin{aligned} C_1(x) &= 1, \\ C_2(x) &= 1, \\ C_{i+2}(x) &= C_{i+1}(x) + x^{2^{ik}} C_i(x) \text{ for } 1 \leq i \leq t-1. \end{aligned} \tag{9}$$

Another polynomial $Z_n(x)$ over \mathbb{F}_{2^n} is defined as follows.

$$\begin{aligned} Z_1(x) &= 1, \\ Z_t(x) &= C_{n+1}(x) + x C_{t-1}^{2^k}(x) \text{ for } t > 1. \end{aligned} \tag{10}$$

Then we have the following result from [HK10, Proposition 2].

Proposition 4. *Let $\gcd(k, n) = d \geq 1$ and $n = td$ for $t > 1$. The equation*

$$\lambda^{2^k} x^{2^{2k}} + x^{2^k} + \lambda x = 0$$

defined over \mathbb{F}_{2^n} has no non zero solution in \mathbb{F}_{2^n} if and only if $Z_t(\lambda) \neq 0$.

The form of λ for which $Z_t(\lambda) = 0$ is known, which is as follows.

Lemma 1. [HK10, Corollary 1] Let $\gcd(k, n) = d \geq 1$ and $n = td$ for $t > 1$. Then α is a zero of $Z_t(x)$ in \mathbb{F}_{2^n} if and only if it is of the form

$$\frac{v_0^{2^{2k}+1}}{(v_0 + v_1)^{2^{k+1}}}, \quad (11)$$

where $v_0 \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^d}$ and $v_1 = v_0^{2^k}$. The total number of distinct roots are

$$\begin{cases} \frac{2^{n+d}-2^d}{2^{2d}-1} & \text{for even } t \\ \frac{2^{n+d}-2^{2d}}{2^{2d}-1} & \text{for odd } t. \end{cases}$$

Therefore, we have the following theorem which characterizes the quadratic negabent monomials.

Theorem 2. The function $f : x \mapsto \text{Tr}(\lambda x^{2^k+1})$ is negabent if and only if λ can not be written as $\frac{v_0^{2^{2k}+1}}{(v_0+v_1)^{2^{k+1}}}$ for $v_0 \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^d}$ and $v_1 = v_0^{2^k}$ where $\gcd(k, n) = d$ and $n = td$.

Proof. Proposition 3 and Proposition 4 imply that f is negabent if and only if λ is not a zero of $Z_t(x)$ where $\gcd(k, n) = d$ and $n = td$. From Lemma 1, we know that $Z_t(\lambda) \neq 0$ if and only if λ is not of the form $\frac{v_0^{2^{2k}+1}}{(v_0+v_1)^{2^{k+1}}}$ where $v_0 \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^d}$ and $v_1 = v_0^{2^k}$. □

3.3 Quadratic bent-negabent functions

We recall the well known result on the quadratic bent monomials. This is directly taken from [DL04].

Lemma 2. [DL04] Let $\lambda \in \mathbb{F}_{2^n}$ and n even. The function $f : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_2$ with

$$f(x) = \text{Tr}(\lambda x^{2^k+1})$$

is bent if and only if

$$\lambda \notin \{x^{2^k+1} \mid x \in \mathbb{F}_{2^n}\}.$$

Note that $\lambda^{2^{n-k}} x^{2^{n-k}} + \lambda x^{2^k}$ is a permutation if and only if $\lambda \notin \{x^{2^k+1} \mid x \in \mathbb{F}_{2^n}\}$. Therefore, if $f(x) = \text{Tr}(\lambda x^{2^k+1})$ is negabent, then Proposition 3 tells that $\lambda^{2^{n-k}} x^{2^{n-k}} + \lambda x^{2^k} + x$ is also a permutation polynomial, i.e., $\lambda^{2^{n-k}} x^{2^{n-k}} + \lambda x^{2^k}$ is a complete mapping polynomial. We summarize these results as follows.

Theorem 3. Let $\lambda \in \mathbb{F}_{2^n}$ where n is even. The function $f : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_2$ with

$$f(x) = \text{Tr}(\lambda x^{2^k+1})$$

is bent negabent if and only if one of the following two equivalent statements holds.

1. $\lambda^{2^{n-k}} x^{2^{n-k}} + \lambda x^{2^k}$ is a complete mapping polynomial.
2. λ is neither of the form $\frac{v_0^{2^{2k+1}}}{(v_0+v_1)^{2^k+1}}$ nor of the form v^{2^k+1} for $v \in \mathbb{F}_{2^n}$, $v_0 \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^d}$ and $v_1 = v_0^{2^k}$, where $\gcd(k, n) = d$ and $n = td$.

The existence of quadratic bent-negabent functions is known [PP07, Theorem 5]. However, we reprove the same result by simple counting argument and using the previous characterization of the bent-negabent functions.

Theorem 4. For all $n \geq 4$, quadratic bent-negabent functions always exist.

Proof. We show that there always exists a $\lambda \in \mathbb{F}_{2^n}$ which satisfies the condition 2 of Theorem 3.

If $\gcd(2^k + 1, 2^n - 1) = 1$, then $x \mapsto x^{2^k+1}$ is a bijection. Then for any $\lambda \in \mathbb{F}_{2^n}$ there exists $x \in \mathbb{F}_{2^n}$ such that $\lambda = x^{2^k+1}$. Therefore, if $\text{Tr}(\lambda x^{2^k+1})$ is bent then $\gcd(2^k + 1, 2^n - 1) > 1$. Since 2 does not divide both of $2^k + 1$ and $2^n - 1$. Therefore, $\gcd(2^k + 1, 2^n - 1) \geq 3$. Let $S_1 = \{x^{2^k+1} | x \in \mathbb{F}_{2^n}\}$, then $|S_1| \leq \frac{2^n-1}{3}$. On the other hand, if $\gcd(k, n) = d$ and $n = td$, then by Lemma 1, we know that the number of possible $\lambda \in \mathbb{F}_{2^n}$ such that λ is of the form $\frac{v_0^{2^{2k+1}}}{(v_0+v_1)^{2^k+1}}$ is $\begin{cases} \frac{2^{n+d}-2^d}{2^{2d}-1} & \text{for even } t \\ \frac{2^{n+d}-2^{2d}}{2^{2d}-1} & \text{for odd } t \end{cases}$. Let $S_2 = \{y \in \mathbb{F}_{2^n} | y = \frac{v_0^{2^{2k+1}}}{(v_0+v_1)^{2^k+1}}\}$. Note that $|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|$. Then

$$|S_1 \cup S_2| \leq \frac{2^n - 1}{3} + \frac{2^{n+d} - 2^d}{2^{2d} - 1} - |S_1 \cap S_2|.$$

Therefore,

$$\begin{aligned} 2^n - |S_1 \cup S_2| &\geq 2^n - \frac{2^n - 1}{3} - \frac{2^{n+d} - 2^d}{2^{2d} - 1} + |S_1 \cap S_2| \\ &= (2^n - 1) \cdot \frac{2 \cdot 2^{2d} - 3 \cdot 2^d - 2}{3(2^{2d} - 1)} + |S_1 \cap S_2| + 1 \\ &\geq |S_1 \cap S_2| + 1, \quad \text{since } 2 \cdot 2^{2d} - 3 \cdot 2^d - 2 \geq 0. \end{aligned}$$

Therefore, we see that a $\lambda \in \mathbb{F}_{2^n}$ always exists that satisfies Condition 2 of Theorem 3.

This proves the theorem. \square

Below we characterize bent-negabent functions when $n = 2k$.

Proposition 5. *Let $n = 2k$ and $f : x \mapsto \text{Tr}(\lambda x^{2^k+1})$ be a quadratic function defined over \mathbb{F}_{2^n} . Then f is negabent if and only if $\lambda + \lambda^{2^k} \neq 1$. Moreover, f is bent-negabent if and only if $\lambda + \lambda^{2^k} \notin \mathbb{F}_2$.*

Proof. By Proposition 3 we have that f is negabent if and only if $P(x) = \lambda^{2^{n-k}} x^{2^{n-k}} + \lambda x^{2^k} + x$ is a permutation, i.e., $P(x)^{2^k} = \lambda x + \lambda^{2^k} x^{2^{2k}} + x^{2^k}$ is a permutation. Since $n = 2k$, therefore, $P(x)^{2^k} = (\lambda + \lambda^{2^k})x + x^{2^k}$. Now $(\lambda + \lambda^{2^k})x + x^{2^k}$ is permutation if and only if $(\lambda + \lambda^{2^k})x + x^{2^k} \neq 0$, i.e., $\lambda + \lambda^{2^k} \neq x^{2^k-1}$, for all $x \in \mathbb{F}_{2^n}^*$. Note that $\lambda + \lambda^{2^k} \in \mathbb{F}_{2^k}$ for all $\lambda \in \mathbb{F}_{2^n}$ and the mapping $\lambda \mapsto \lambda + \lambda^{2^k}$ is onto. Let us consider the group $G = \{x^{2^k-1} | x \in \mathbb{F}_{2^n}^*\}$. The intersection of \mathbb{F}_{2^k} and G is $\{1\}$. Therefore, $\text{Tr}(\lambda x^{2^k+1})$ is negabent if and only if $\lambda + \lambda^{2^k} \neq 1$. We know that f is bent if and only if $\lambda \neq x^{2^k+1}$ for some $x \in \mathbb{F}_{2^n}$. Note that if $\lambda = x^{2^k+1}$ then $\lambda \in \mathbb{F}_{2^k}$ and $\lambda + \lambda^{2^k} = 0$. Therefore, f is bent-negabent if and only if $\lambda + \lambda^{2^k} \notin \mathbb{F}_2$. \square

4 Maiorana-McFarland bent-negabent functions

Maiorana-McFarland is an important class of bent functions which was extensively studied by Dillon [Dil74, pp. 90-95]. This class is usually called the *class* \mathcal{M} of bent functions.

Lemma 3. *Let $n = 2t$. Let us consider a Boolean function f defined by*

$$f : (x, y) \in \mathbb{F}_{2^t} \times \mathbb{F}_{2^t} \mapsto \text{Tr}_1^t(x\pi(y) + h(y)) \quad (12)$$

where π is a function over \mathbb{F}_{2^t} and h is any function on \mathbb{F}_{2^t} . Then f is a bent function if and only if π is a bijection.

Theorem 5. *Let f be a Maiorana-McFarland function as in Lemma 3. Then f is negabent if and only if for all $a, b \in \mathbb{F}_{2^t}^*$*

$$\sum_{y \in Y_{a,b}} (-1)^{\text{Tr}_1^t(a\pi(y)) + h(y) + h(y+b) + by} = 0, \quad (13)$$

where $Y_{a,b} = \{y \in \mathbb{F}_{2^t} | \pi(y) + \pi(y+b) = a\}$ such that $Y_{a,b}$ is non empty.

Proof. From (5) we have, $f(x, y)$ is negabent if and only if for all $(a, b) \in \mathbb{F}_{2^t} \times \mathbb{F}_{2^t} \setminus \{(0, 0)\}$

$$\begin{aligned} \sum_{(x,y) \in \mathbb{F}_{2^t} \times \mathbb{F}_{2^t}} (-1)^{f(x,y)+f(x+a,y+b)+Tr_1^t(ax)+Tr_1^t(by)} &= 0 \\ \sum_{(x,y) \in \mathbb{F}_{2^t} \times \mathbb{F}_{2^t}} (-1)^{Tr_1^t(x(\pi(y)+\pi(y+b)+a))+Tr_1^t(a\pi(y+b)+h(y)+h(y+b)+by)} &= 0. \end{aligned}$$

Let

$$S_{a,b} = \sum_{(x,y) \in \mathbb{F}_{2^t} \times \mathbb{F}_{2^t}} (-1)^{Tr_1^t(x(\pi(y)+\pi(y+b)+a))+Tr_1^t(a\pi(y+b)+h(y)+h(y+b)+by)}.$$

We treat the sum $S_{a,b}$ in the following cases.

CASE 1: For $a \neq 0$ and $b = 0$. Then

$$\begin{aligned} S_{a,b} &= \sum_{(x,y) \in \mathbb{F}_{2^t} \times \mathbb{F}_{2^t}} (-1)^{Tr_1^t(ax)+Tr_1^t(a\pi(y))} \\ &= \sum_{x \in \mathbb{F}_{2^t}} (-1)^{Tr_1^t(ax)} \sum_{y \in \mathbb{F}_{2^t}} (-1)^{Tr_1^t(a\pi(y))} \\ &= 0. \end{aligned}$$

CASE 2: For $a = 0$ and $b \neq 0$. Then

$$\begin{aligned} S_{a,b} &= \sum_{(x,y) \in \mathbb{F}_{2^t} \times \mathbb{F}_{2^t}} (-1)^{Tr_1^t(x(\pi(y)+\pi(y+b))) + Tr_1^t(h(y)+h(y+b)+by)} \\ &= \sum_{y \in \mathbb{F}_{2^t}} (-1)^{Tr_1^t(h(y)+h(y+b)+by)} \sum_{x \in \mathbb{F}_{2^t}} (-1)^{Tr_1^t(x(\pi(y)+\pi(y+b)))} \\ &= \sum_{y \in \mathbb{F}_{2^t}} (-1)^{Tr_1^t(h(y)+h(y+b)+by)} \times 0 \\ &\quad \text{since } \pi \text{ is a permutation, } \pi(y) \neq \pi(y+b) \\ &= 0. \end{aligned}$$

CASE 3: For $a \neq 0$ and $b \neq 0$. Then

$$\begin{aligned} S_{a,b} &= \sum_{(x,y) \in \mathbb{F}_{2^t} \times \mathbb{F}_{2^t}} (-1)^{Tr_1^t(a\pi(y+b)+h(y)+h(y+b)+by)+Tr_1^t(x(\pi(y)+\pi(y+b)+a))} \\ &= \sum_{y \in \mathbb{F}_{2^t}} (-1)^{Tr_1^t(a\pi(y+b)+h(y)+h(y+b)+by)} \sum_{x \in \mathbb{F}_{2^t}} (-1)^{Tr_1^t(x(\pi(y)+\pi(y+b)+a))}. \end{aligned}$$

If there exists some y such that $y \notin Y_{a,b}$, i.e., $\pi(y) + \pi(y+b) \neq a$, then

$$\sum_{x \in \mathbb{F}_{2^t}} (-1)^{Tr_1^t(x(\pi(y) + \pi(y+b) + a))} = 0.$$

On the other hand if $y \in Y_{a,b}$, i.e., $\pi(y) + \pi(y+b) = a$, then

$$\sum_{x \in \mathbb{F}_{2^t}} (-1)^{Tr_1^t(x(\pi(y) + \pi(y+b) + a))} = 2^t.$$

Therefore,

$$\begin{aligned} S_{a,b} &= 2^t \sum_{y \in Y_{a,b}} (-1)^{Tr_1^t(a\pi(y+b) + h(y) + h(y+b) + by)} \\ &= 2^t \sum_{y \in Y_{a,b}} (-1)^{Tr_1^t(a\pi(y) + a^2 + h(y) + h(y+b) + by)} \\ &\quad \text{since } \pi(y+b) = \pi(y) + a \text{ for } y \in Y_{a,b} \\ &= 2^t (-1)^{Tr(a^2)} \sum_{y \in Y_{a,b}} (-1)^{Tr_1^t(a\pi(y) + h(y) + h(y+b) + by)}. \end{aligned}$$

Therefore, $S_{a,b} = 0$ if and only if

$$\sum_{y \in Y_{a,b}} (-1)^{Tr_1^t(a\pi(y) + h(y) + h(y+b) + by)} = 0.$$

Thus after discussing all the above cases it is clear that the Maiorana-McFarland bent function f is negabent if and only if

$$\sum_{y \in Y_{a,b}} (-1)^{Tr_1^t(a\pi(y) + h(y) + h(y+b) + by)} = 0.$$

□

This Theorem gives us the clue to construct negabent functions over the finite fields that belong to the class of Maiorana-McFarland bent functions.

Definition 2. A mapping $F : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_{2^n}$ is called homomorphic if $F(x+y) = F(x) + F(y)$ and $F(xy) = F(x)F(y)$ for all $x, y \in \mathbb{F}_{2^n}$.

The only possible homomorphic permutation over \mathbb{F}_{2^n} is of the form $x \mapsto x^{2^i}$. Note that $Tr_1^n(x) = Tr_1^n(x^{2^i})$, therefore the mapping $x \mapsto Tr_1^n(x)$ is invariant under the action of this permutation. Using this observation we show an interesting consequence of Theorem 5, when the permutation π is chosen as $\pi(x) = x^{2^i}$.

Theorem 6. Let $f : (x, y) \in \mathbb{F}_{2^t} \times \mathbb{F}_{2^t} \mapsto \mathbb{F}_2$ be a Maiorana-McFarland bent function given by

$$f(x, y) = \text{Tr}_1^t(xy^{2^i} + h(y)), \quad (14)$$

Then f is negabent if and only if $\text{Tr}_1^t(h(y))$ is a bent function over \mathbb{F}_{2^t} .

Proof. Let $\pi(y) = y^{2^i}$. Then $\pi(y)$ is a homomorphic permutation polynomial over \mathbb{F}_{2^t} . From the linearity of π we have $\pi(y) + \pi(y + b) = a$ if and only if $\pi(b) = a$. Then

$$Y_{a,b} = \begin{cases} \mathbb{F}_{2^t} & \text{when } \pi(b) = a \\ \text{empty} & \text{when } \pi(b) \neq a. \end{cases}$$

Since π is a permutation, for each a there will be a b such that $\pi(b) = a$. For such a and b

$$\begin{aligned} \sum_{y \in \mathbb{F}_{2^t}} (-1)^{\text{Tr}_1^t(a\pi(y) + h(y) + h(y+b) + by)} &= \sum_{y \in \mathbb{F}_{2^t}} (-1)^{\text{Tr}_1^t(\pi(b)\pi(y) + by + h(y) + h(y+b))} \\ &= \sum_{y \in \mathbb{F}_{2^t}} (-1)^{\text{Tr}_1^t(\pi(by)) + \text{Tr}_1^t(by) + \text{Tr}_1^t(h(y) + h(y+b))}. \end{aligned}$$

Note that $\text{Tr}_1^t(y) = \text{Tr}_1^t(\pi(y))$, for all $y \in \mathbb{F}_{2^t}$. So

$$\sum_{y \in \mathbb{F}_{2^t}} (-1)^{\text{Tr}_1^t(a\pi(y) + h(y) + h(y+b) + by)} = \sum_{y \in \mathbb{F}_{2^t}} (-1)^{\text{Tr}_1^t(h(y) + h(y+b))}.$$

Using Theorem 5, the function f is negabent if and only if

$$\sum_{y \in \mathbb{F}_{2^t}} (-1)^{\text{Tr}_1^t(h(y)) + \text{Tr}_1^t(h(y+b))} = 0,$$

for all $b \in \mathbb{F}_{2^t}^*$, i.e., $\text{Tr}_1^t(h(y))$ is a bent function over \mathbb{F}_{2^t} .

Thus the result follows. \square

Similar kind of result was proved in [SGC⁺12], where the function was defined over the vector space \mathbb{F}_2^n and the permutation was such that $wt(x+y) = wt(\pi(x) + \pi(y))$. However, the result of Theorem 6 is quite distinct as it is in the domain of finite fields. Moreover, Theorem 5 is a general characterization of bent-negabent Maiorana-McFarland functions and several constructions of Maiorana-McFarland bent-negabent functions can be obtained from this. For instance, Theorem 6 allows us to construct bent-negabent Maiorana-McFarland function of degree $n/4$ over \mathbb{F}_{2^n} by choosing a bent function of degree $n/4$ as h , where $n = 2t$.

5 Negabent functions from bent functions

We show that given a negabent function over a finite field, one can construct a bent function, and vice versa. First we define $Q : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_2$ as

$$Q(x) = \sum_{i=1}^{\frac{n}{2}-1} \text{Tr}_1^n(x^{2^i+1}) + \text{Tr}_1^{\frac{n}{2}}(x^{2^{\frac{n}{2}}+1}) \quad (15)$$

As mentioned earlier, for simplicity we write $\text{Tr}_1^n(x) = \text{Tr}(x)$.

We also mention a result from [CC03] and [CCCF01] which will be useful in proving our result. These results were proved for Boolean functions defined over vector spaces, however, it is easy to see the equivalent results when the Boolean function is defined by the trace representation.

Lemma 4. [CC03, Corollary 1] Suppose $H_\beta = \{x \in \mathbb{F}_{2^n} : \text{Tr}(\beta x) = 0\}$ is a hyperplane. If $f : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_2$ is bent, then for any $a \notin H_\beta$,

$$\sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x)+f(x+a)+\text{Tr}(\beta x)} = 0. \quad (16)$$

Lemma 5. [CCCF01, Theorem V.3] The Boolean function $f : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_2$ is bent if and only if there exists a hyperplane \mathcal{H} such that $f(x) + f(x+a)$ is balanced for every nonzero $a \in \mathcal{H}$.

Theorem 7. Suppose $f : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_2$, and Q is as defined in (15).

1. if f is bent then $f + Q$ is negabent.
2. If f is negabent then $f + Q$ is bent,

Proof. Suppose $a \in F_{2^n}^*$, then

$$\begin{aligned}
Q(x) + Q(x+a) &= \sum_{i=1}^{\frac{n}{2}-1} \text{Tr}(a^{2^i}x + ax^{2^i}) + \text{Tr}_1^{\frac{n}{2}}(ax^{2^{\frac{n}{2}}} + a^{2^{\frac{n}{2}}}x) + \text{constant} \\
&= \sum_{i=1}^{\frac{n}{2}-1} \text{Tr}((a^{2^i} + a^{2^{n-i}})x) + \text{Tr}_1^n(a^{2^{\frac{n}{2}}}x) + \text{constant} \\
&= \sum_{i=1}^{\frac{n}{2}-1} \text{Tr}((a^{2^i} + a^{2^{n-i}})x) + \text{Tr}(a^{2^{\frac{n}{2}}}x) + \text{Tr}(ax) \\
&\quad + \text{Tr}(ax) + \text{constant}, \\
&= \text{Tr}(\text{Tr}(a)x) + \text{Tr}(ax) + \text{constant} \\
&= \text{Tr}(a)\text{Tr}(x) + \text{Tr}(ax) + \text{constant}. \tag{17}
\end{aligned}$$

Without any loss of generality we ignore the constant term in $Q(x)+Q(x+a)$.

Case 1: We prove that if f is bent, then $f + Q$ is negabent, for which we have to show that

$$\sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x)+f(x+a)+Q(x)+Q(x+a)+\text{Tr}(ax)} = 0$$

We have

$$\begin{aligned}
&\sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x)+f(x+a)+Q(x)+Q(x+a)+\text{Tr}(ax)} \\
&= \sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x)+f(x+a)+\text{Tr}(a)\text{Tr}(x)}.
\end{aligned}$$

Subcase 1.1: If $\text{Tr}(a) = 0$, then

$$\sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x)+f(x+a)+\text{Tr}(a)\text{Tr}(x)} = 0, \text{ since } f \text{ is bent.}$$

Subcase 1.2: If $\text{Tr}(a) = 1$, then a does not belong to the hyperplane

$$H_1 = \{x \in \mathbb{F}_{2^n} : \text{Tr}(1.x) = 0\}.$$

Therefore, by Lemma 4,

$$\sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x)+f(x+a)+Tr(1.x)} = 0.$$

So for any $a \in F_{2^n}^*$,

$$\sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x)+f(x+a)+Q(x)+Q(x+a)+Tr(ax)} = 0.$$

This implies that $f + Q$ is negabent.

Case 2: Next we suppose that f is negabent and prove that $g = f + Q$ is bent. For any $a \in \mathbb{F}_{2^n}^*$ we have

$$\begin{aligned} & \sum_{x \in \mathbb{F}_{2^n}} (-1)^{g(x)+g(x+a)} \\ &= \sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x)+f(x+a)+Q(x)+Q(x+a)} \\ &= \sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x)+f(x+a)+Tr(a)Tr(x)+Tr(ax)}, \text{ by (17)} \\ &= \sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x)+f(x+a)+Tr(ax)}, \text{ if } a \in H_1, \text{ i.e. } Tr(a) = 0 \\ &= 0, \text{ since } f \text{ is negabent.} \end{aligned}$$

Therefore, we see that for any nonzero a that belongs to the hyperplane H_1 , $g(x) + g(x+a)$ is balanced. Hence by Lemma 5, we prove that g is bent. \square

This theorem has interesting consequences.

Corollary 1. *The Boolean function $f : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_2$ is bent-negabent if and only if both f and $f + Q$ are bent.*

Corollary 2. *The Boolean function $f : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_2$ is bent-negabent if and only if $f + Q$ is bent-negabent.*

Corollary 3. *The function Q is bent but not negabent.*

Proof. It is easy to check that Q is bent by looking at its derivative given in (17).

Now on the contrary, assume that Q is negabent. Then $g = Q + Q = 0$ is bent (by Theorem 7), which is a contradiction. \square

We now use the result of Theorem 7 to construct bent-negabent functions. Note that any two quadratic bent functions are affine equivalent. It is clear that there is one-one correspondence between the bent function defined over $\mathbb{F}_{2^{2t}}$ and $\mathbb{F}_{2^t} \times \mathbb{F}_{2^t}$. With abuse of notation we use $Q(x, y)$ defined over $\mathbb{F}_{2^t} \times \mathbb{F}_{2^t}$ as the corresponding bent function for $Q(x)$ which is defined in (15). That means the bent function $G : (x, y) \in \mathbb{F}_{2^t} \times \mathbb{F}_{2^t} \mapsto \mathbb{F}_2$ given by

$$G(x, y) = Tr_1^t(xy), \quad (18)$$

is affine equivalent to the bent function $Q(x, y)$. This also means by Theorem 7 that if $f(x, y)$ is a bent function then $f(x, y) + Q(x, y)$ is negabent and vice versa.

Suppose $G(x, y)$ and $Q(x, y)$ are related by the relation

$$Q(x, y) = G(\alpha_1 x + \alpha_2, \alpha_3 y + \alpha_4) + Tr_1^t(\beta x) + Tr_1^t(\gamma y) + c, \quad (19)$$

for some $\alpha_1, \alpha_2, \alpha_3, \alpha, \beta, \gamma$ in \mathbb{F}_2^t , $c \in \mathbb{F}_2$.

Theorem 8. *Let $f : (x, y) \in \mathbb{F}_{2^t} \times \mathbb{F}_{2^t} \mapsto \mathbb{F}_2$ be a Maiorana-McFarland bent function given by*

$$f(x, y) = Tr_1^t(x\pi(y)) + Tr_1^t(h(y)),$$

where $\pi(y)$ is a complete mapping polynomial over \mathbb{F}_{2^t} , $h(y)$ is any polynomial over \mathbb{F}_{2^t} , and $G : (x, y) \in \mathbb{F}_{2^t} \times \mathbb{F}_{2^t} \mapsto \mathbb{F}_2$ defined by $G(x, y) = Tr_1^t(xy)$. Then

$$\begin{aligned} F(x, y) &= f(\alpha_1 x + \alpha_2, \alpha_3 y + \alpha_4) + G(\alpha_1 x + \alpha_2, \alpha_3 y + \alpha_4) \\ &\quad + Tr_1^t(\beta x) + Tr_1^t(\gamma y) + c \end{aligned} \quad (20)$$

is a bent-negabent function.

Proof. We have

$$\begin{aligned} f(x, y) + G(x, y) &= Tr_1^t(x\pi(y)) + Tr_1^t(h(y)) + Tr_1^t(xy) \\ &= Tr_1^t(x(\pi(y) + y)) + Tr_1^t(h(y)). \end{aligned}$$

Since $\pi(y)$ is a complete mapping polynomial over \mathbb{F}_{2^t} , $(\pi(y) + y)$ is a permutation polynomial, so $f + G$ is a bent function. This also implies that

$$F(x, y) = f(\alpha_1 x + \alpha_2, \alpha_3 y + \alpha_4) + G(\alpha_1 x + \alpha_2, \alpha_3 y + \alpha_4) + Tr_1^t(\beta x) + Tr_1^t(\gamma y) + c$$

is a bent function. We have

$$\begin{aligned}
& F(x, y) \\
&= f(\alpha_1 x + \alpha_2, \alpha_3 y + \alpha_4) + G(\alpha_1 x + \alpha_2, \alpha_3 y + \alpha_4) + \text{Tr}_1^t(\beta x) + \text{Tr}_1^t(\gamma y) + c \\
&= f(\alpha_1 x + \alpha_2, \alpha_3 y + \alpha_4) + Q(x, y), \text{ by 19.}
\end{aligned}$$

Note that $F(x, y) + Q(x, y) = f(\alpha_1 x + \alpha_2, \alpha_3 y + \alpha_4)$ is also bent. So both $F(x, y)$ and $F(x, y) + Q(x, y)$ are bent. Therefore, by Corollary 1, $F(x, y)$ is bent-negabent. \square

At this point, we would like to refer to [SGC⁺12, Theorem 22], which also states a result that is similar to Theorem 8. In that result, the Boolean function is defined over the vector space \mathbb{F}_2^n . Note that complete mapping polynomials are defined over finite fields, however, the proof of [SGC⁺12, Theorem 22] works in the vector space domain. They claim that $\pi(x_1, \dots, x_t)$ is a permutation of \mathbb{F}_2^t that corresponds to the permutation $\pi(x)$ over the field \mathbb{F}_{2^t} , as well as $\pi(x_1, \dots, x_t) \oplus (x_1, \dots, x_t)$ is the permutation of \mathbb{F}_2^t that corresponds to the permutation $\pi(x) + x$ over the field \mathbb{F}_{2^t} . But it is not clear how this correspondence is realized. On the other hand, Theorem 8 can directly apply the complete mapping polynomials as the underlying Boolean function is defined over a finite field.

Now we construct infinite classes of n -variable bent-negabent function with the maximum degree $\frac{n}{2}$. Our construction is similar to that of Theorem 5 of [SPT13]. Their proof works when there is a permutation polynomial $p(x)$ over the vector space \mathbb{F}_2^n such that $p(x) + x$ is also a permutation polynomial over \mathbb{F}_2^n . However, this kind of permutation over the vector space \mathbb{F}_2^n is not characterized, on the other hand, these kind of permutation polynomials (complete mapping polynomials) are well characterized over finite field.

Corollary 4. *Suppose $n = 2t$. Then the n -variable function $F(x, y)$ defined in Theorem 8, where the polynomial $h(y)$ has algebraic degree $t = \frac{n}{2}$ is a bent-negabent function of degree $\frac{n}{2}$.*

Proof. The algebraic degree of $h(y)$ is $t = \frac{n}{2}$, which implies that the degree of $f(x, y) + G(x, y)$ is also $\frac{n}{2}$. That also implies that the degree $F(x, y) = f(\alpha_1 x + \alpha_2, \alpha_3 y + \alpha_4) + G(\alpha_1 x + \alpha_2, \alpha_3 y + \alpha_4) + \text{Tr}_1^t(\beta x) + \text{Tr}_1^t(\gamma y) + c$ is $\frac{n}{2}$. \square

Two infinite classes of complete mapping polynomial are given in [LC07].

Theorem 9. [LC07, Theorem 4.3] Let p be a prime and m and ℓ are two positive integers. Let k be the multiplicative order of p in \mathbb{Z}_m . Assume $a \in \mathbb{F}_{p^{k\ell}}$ is such that $(-a)^m \neq 1$. Then the polynomials

$$\pi_1(x) = x(x^{\frac{p^{k\ell m}-1}{m}} + a),$$

and

$$\pi_2(x) = ax^{\frac{p^{k\ell m}-1}{m}+1},$$

are complete mapping polynomials over $\mathbb{F}_{p^{k\ell m}}$.

Theorem 10. Suppose $n = 2k\ell m$ and $\pi_1(y), \pi_2(y)$ are the complete mapping polynomials as given in Theorem 9. Then the n -variable function $F(x, y)$ as given in (20) is a bent-negabent with degree $\frac{n}{2}$, for $f(x, y) = \text{Tr}_1^{\frac{n}{2}}(x\pi_1(y)) + \text{Tr}_1^{\frac{n}{2}}(y^{2^{\frac{n}{2}}-1})$ and $f(x, y) = \text{Tr}_1^{\frac{n}{2}}(x\pi_2(y)) + \text{Tr}_1^{\frac{n}{2}}(y^{2^{\frac{n}{2}}-1})$.

Proof. This follows easily as the algebraic degree of $\text{Tr}_1^{\frac{n}{2}}(y^{2^{\frac{n}{2}}-1})$ is $\frac{n}{2}$. \square

6 Conclusion

We have presented some characterizations of negabent functions over the finite field. The analysis done here is useful in order to obtain further results on negabent functions over finite fields. In this paper, we have characterized quadratic negabent monomials. The characterization of negabent monomials of higher degree will be interesting. We also have characterized negabent functions which are Maiorana-McFarland bent. Moreover, we have presented a construction of bent-negabent functions with optimal degree. This is the second known construction of such functions. However, it is interesting to see further classes of such functions.

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